Exploring a family of Kleinian groups— Mumford's discreteness problem

Women in Mathematics

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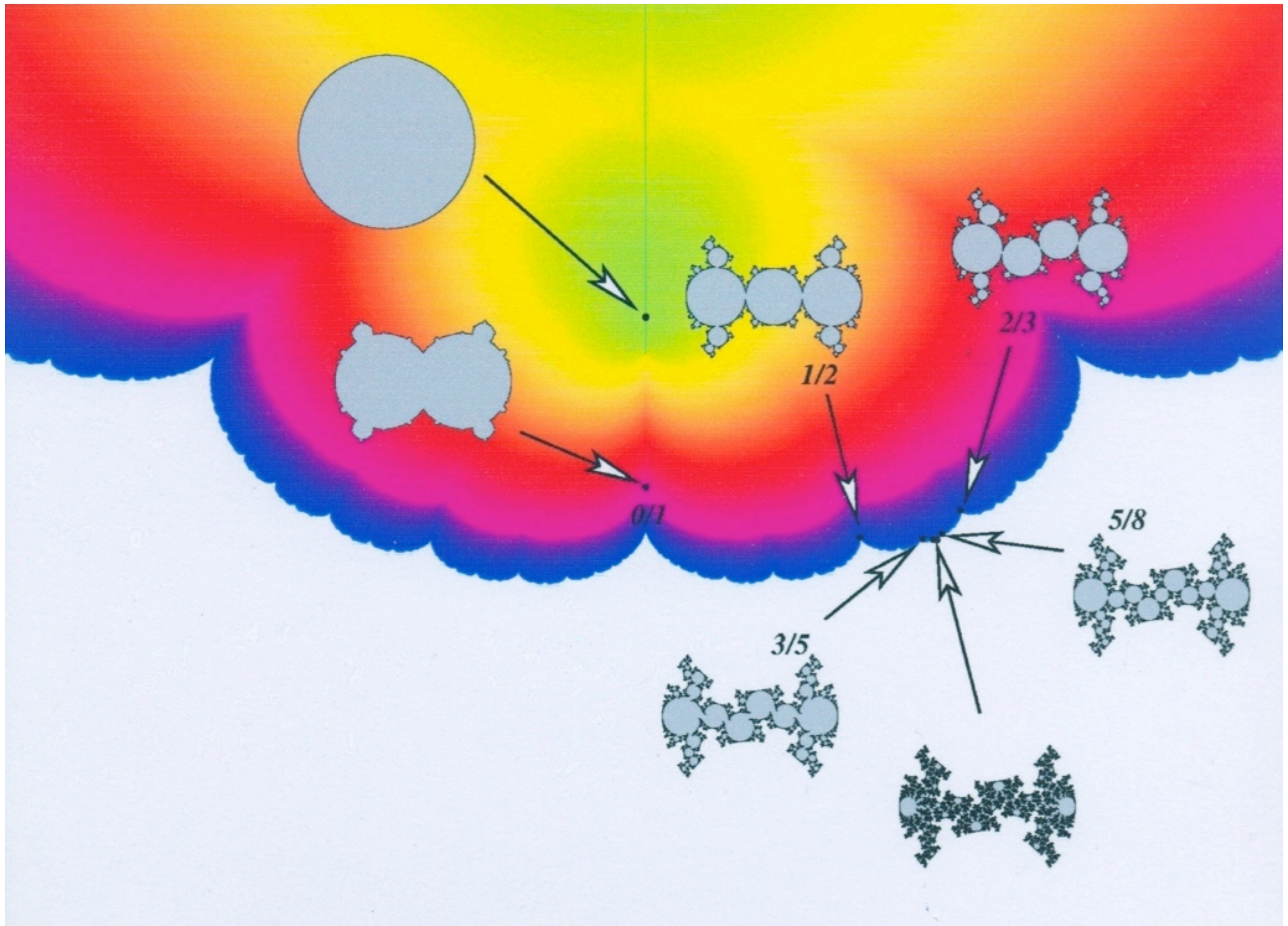
Caroline Series

THE UNIVERSITY OF WARWICK



A parameter space of discrete groups.
Picture by David Wright

Mumford's problem

Inspired by some of the first pictures of the Mandelbrot set, around 1979/80 David Mumford suggested the following problem:

Let G_c be the group generated by two transformations $A : z \mapsto c + 1/z, B \mapsto z + 2$, $z \in \mathbb{C} \cup \infty, c \in \mathbb{C}$. Investigate for which values of c the group G_c is free and discrete.

This is closely analogous to iterating $z \mapsto z^2 + c$.

There are various different possible outcomes, depending on c, as recorded in the famous Mandelbrot set.

David Mumford
Fields Medal 1974
Wolf Prize 2008

Mumford's maps are examples of *Möbius maps,* that is maps of the form $z \mapsto (az + b)/(cz + d)$ from the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ to itself.

These are the most general conformal automorphisms of $\hat{\mathbb{C}}$.

The problem

For which values of $c \in \mathbb{C}$ is the group G_c generated by $A_c : z \mapsto c + 1/z$ and $B : z \mapsto z + 2$ free and discrete?

Discrete means that the group G_c has no accumulation points in the set $SL(2,\mathbb{C})$ of 2×2 matrices with determinant 1. (This is almost the same as saying that there are regions in $\mathbb{C} \cup \infty$ where G_c acts properly discontinuously.)

Free means there are no non-trivial relations.

A discrete group of Möbius maps is called a *Kleinian group*.

We will investigate this problem in two ways:

- using $3D$ hyperbolic geometry
- using Riemann surfaces and Teichmüller theory

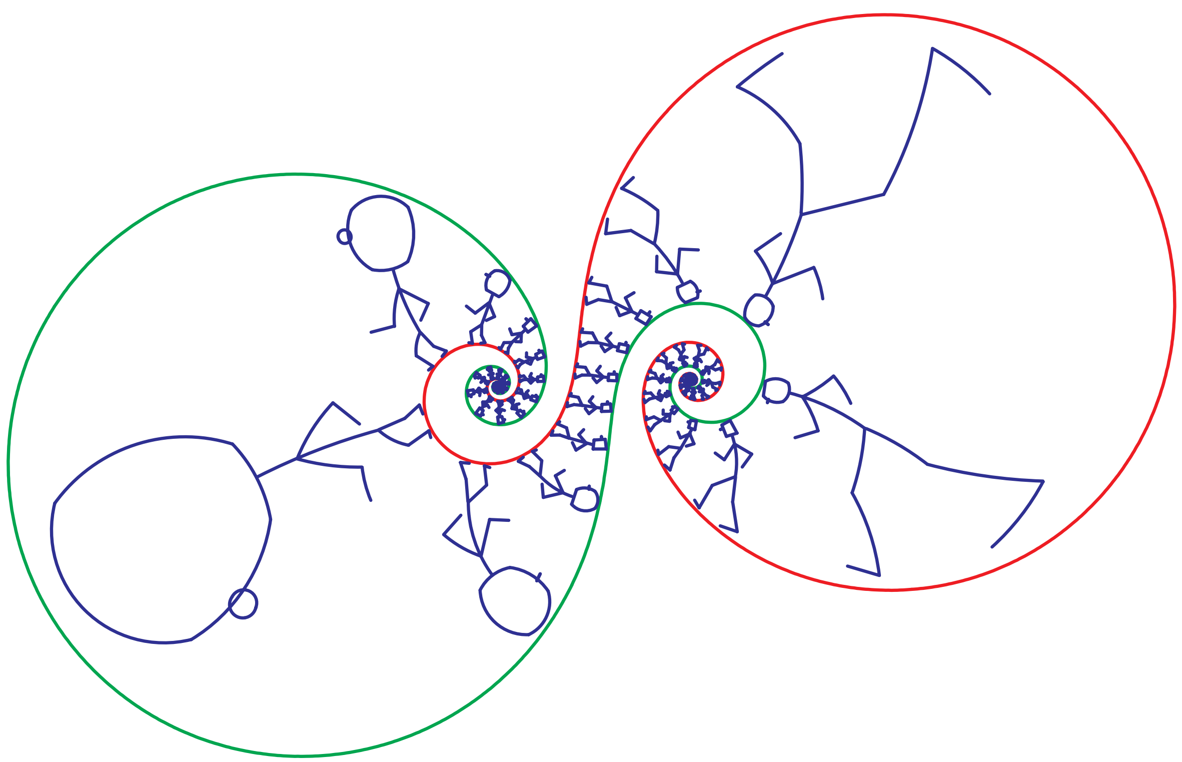
The solution results from the interplay of the two approaches.

First we look at Möbius maps and limit sets of Kleinian groups in a bit more detail.

Möbius maps

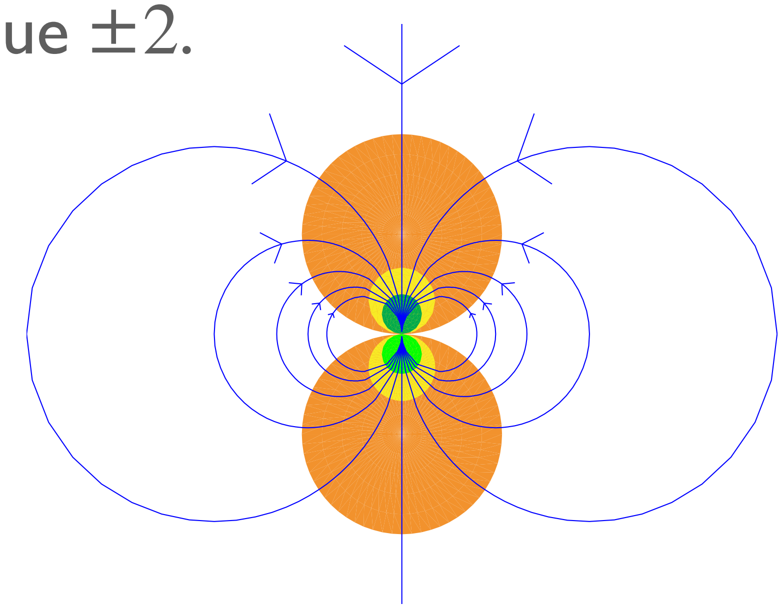
A Möbius map $z \mapsto (az+b)/(cz+d)$ can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ normalized so that $ad - bc = 1$.

With this normalization, Möbius maps are classified dynamically (up to conjugacy) by their Trace $a + d$.

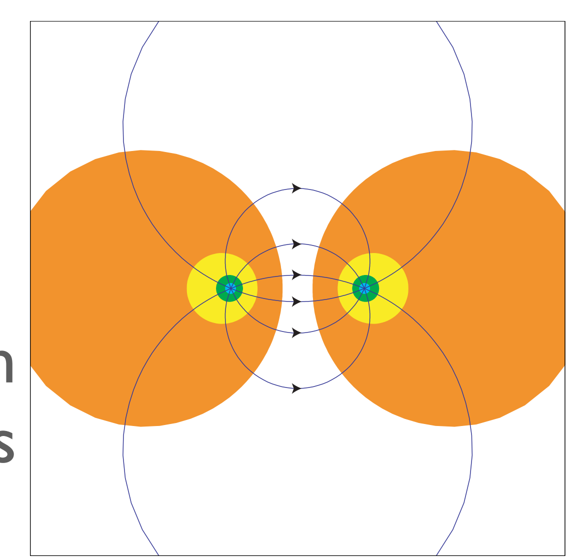


Iterating a typical Möbius map

A Möbius map typically has 2 fixed points. Sometimes the fixed points come together: such maps are called parabolic. This happens only when the trace takes the value ± 2.



Parabolic transformations will become very important later.

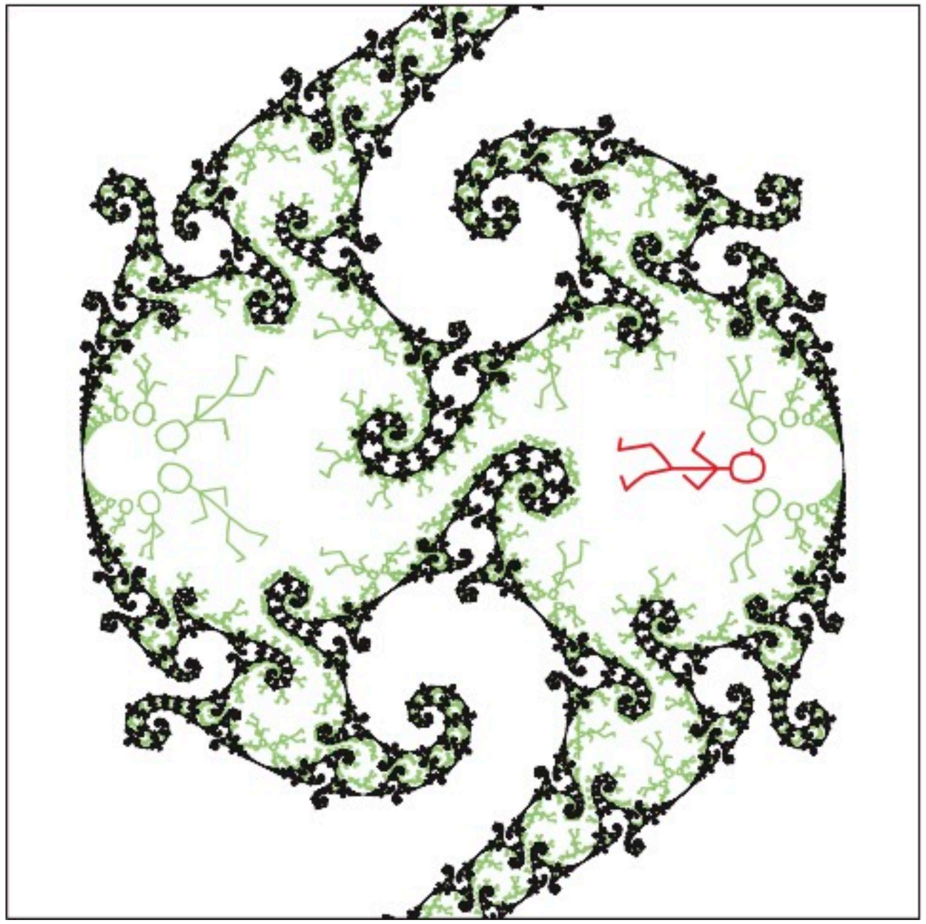


If the trace is real valued then the map doesn't spiral. This will also matter to us later.

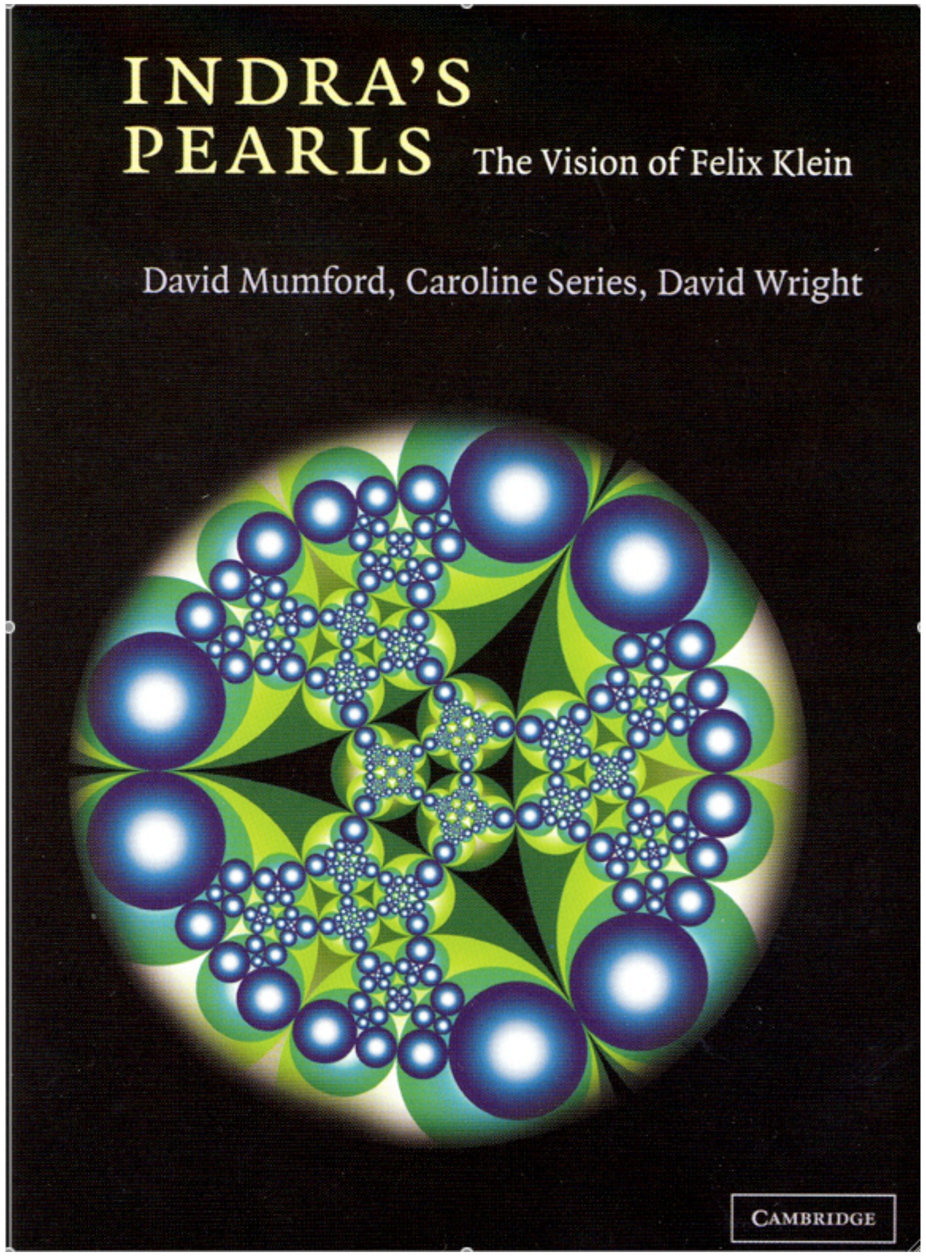
Groups and their limit sets

Unlike Euclidean translations, Möbius maps typically do not commute. In fact they interact in very complicated ways.

Starting with any initial object, its images under a group of Möbius maps get smaller and smaller, piling up on a set called its *limit set*.



Orbits accumulating on the limit set.

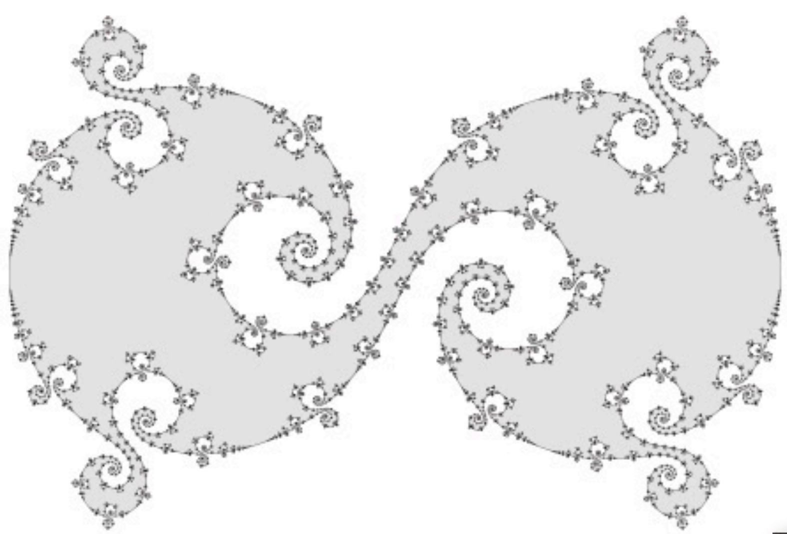


Möbius maps and limit sets are explored in detail in Indra's Pearls. Japanese translation by Yohei Komori (Waseda Univ.)

Discreteness and Möbius maps

Indra's Pearls describes an algorithm to plot limit sets. It will fail if the group in question is not discrete. In this case orbits are dense in $\hat{\mathbb{C}}$.

In these pictures the algorithm plots all fixed points of all elements in the group $G(A, B)$ generated by two maps A and B. This particular system depends on two parameters: t_a = Trace A and t_b = Trace B.



$t_a = 1.91 - 0.05i, t_b = 2$

$t_a = 1.905 - 0.05i, t_b = 2$

If the algorithm here was allowed to run longer, it would fill up the whole page.

Notice that t_a differs by only 0.005 while t_b is the same. So why is $G(A, B)$ discrete for one set of values and not for the other?

Herein lies the difficulty and subtly of Mumford's question.

Approaches to Mumford's problem

Recall the problem: for which values of $c \in \mathbb{C}$ is the group G_c generated by $A_c : z \mapsto c + 1/z$ and $B : z \mapsto z + 2$ free and discrete?

There are two different ways of looking at Möbius maps:

- As conformal automorphisms of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.
- As isometries of $3D$ hyperbolic space $\mathbb{H}^3$.

The corresponding approaches are:

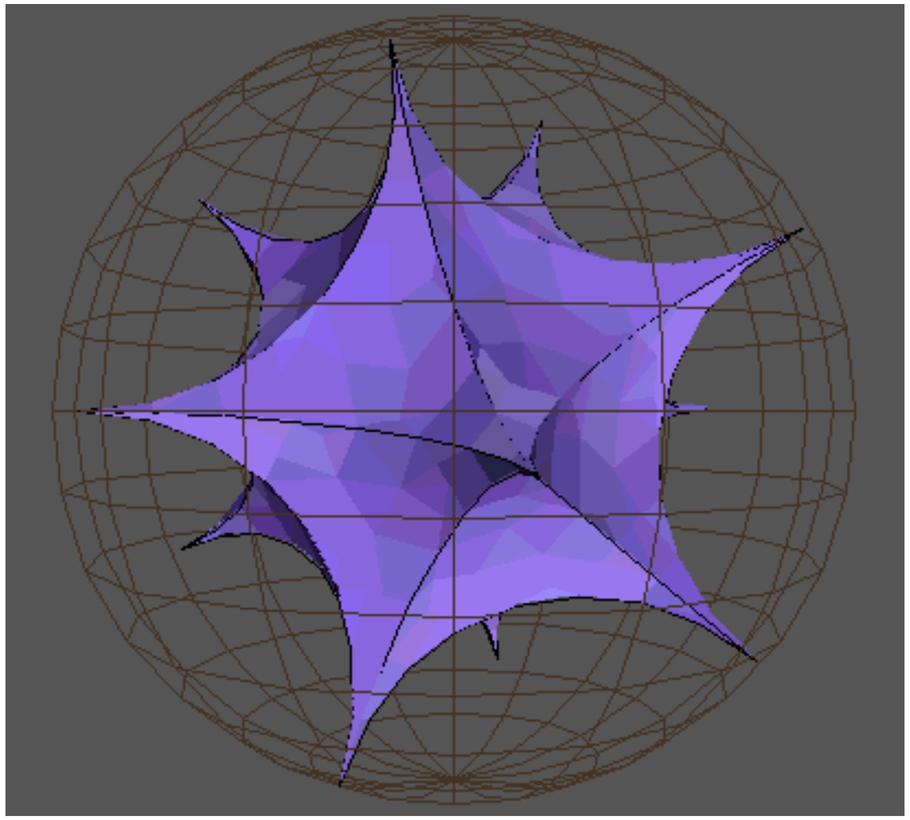
- Through Riemann surfaces and Teichmüller theory as established by Lars Ahlfors and Lipman Bers in the 1960s.

- Through hyperbolic 3-manifolds using ideas introduced by William Thurston in the 1970s-80s.

We'll start with the hyperbolic 3-space approach.

Approach via 3D hyperbolic geometry

Hyperbolic 3-space $\mathbb{H}^3$ can be modelled by the interior of a unit ball B with a suitable metric. Geodesics are circular arcs orthogonal to ∂B and planes are pieces of sphere likewise orthogonal.



Picture from the Geometry Center, Minnesota

∂B is infinitely far, in terms of hyperbolic distance, from the centre, and is often referred to as $\partial\mathbb{H}^3$. It can be identified with the Riemann sphere $\hat{\mathbb{C}}$.

The action of a Möbius map on $\hat{\mathbb{C}}$ extends (using inversions) to an action on the interior of the ball, that is, on $\mathbb{H}^3$. This action is an *isometry* of $\mathbb{H}^3$.

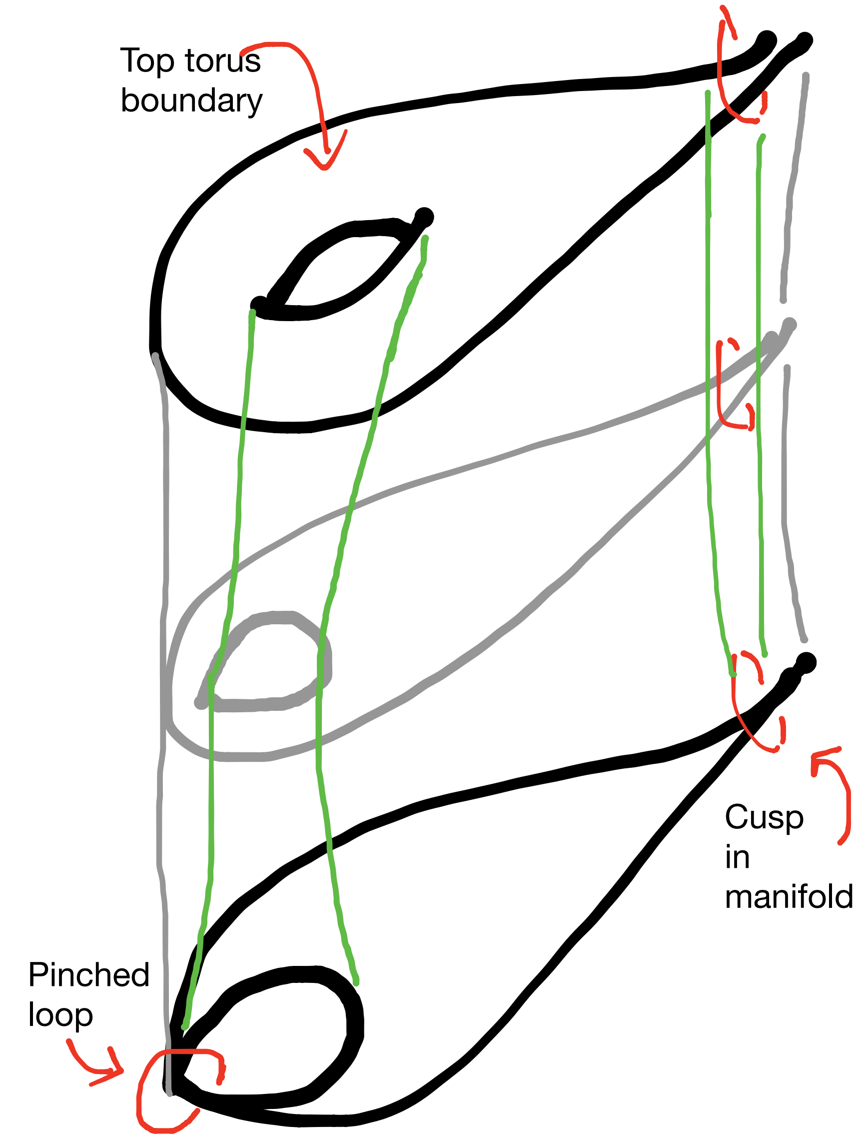
If a group of Möbius maps is discrete and free, then the quotient $\mathbb{H}^3/G$ (made by identifying all points in one G orbit to a single point) is a *hyperbolic 3-manifold,* meaning that small neighbourhoods in $\mathbb{H}^3/G$ are isometric to small balls in $\mathbb{H}^3$.

This gives another perspective on discrete groups of Möbius maps: we can use $3D$ geometry and topology.

Mumford's problem: The hyperbolic 3-manifold

If G_c is the group generated by the maps $A(z) = c + 1/z, B(z) = z + 2$ then what is the hyperbolic 3-manifold $M = \mathbb{H}^3/G_c$?

Two commuting elements A and B would generate a torus T. However A and B don't commute, in fact the commutator $ABA^{-1}B^{-1}$ of A and B is parabolic. This means that the corresponding loop has zero length. Thus the torus contains a puncture, or, in hyperbolic space, a cusp going out to infinity with a missing point at the end. Denote the punctured torus T^*.



The hyperbolic manifold M associated to the group G_c. Elements of the group correspond to curves in M. If the element is parabolic then the curve is pinched to a point.

If G_c is both *discrete* and *free*, we deduce that $M = \mathbb{H}^3/G_c$ is the product $T^* \times \mathbb{R}$. It is an infinite cylinder in which every horizontal section is a copy of T^*. There is an additional pinch point on the bottom boundary, corresponding to the loop B being parabolic.

The `top' and 'bottom' surfaces

The surfaces at the top and bottom of $M = \mathbb{H}^3/G_c$ are infinitely far from the 'middle' of M. They form its 'boundary at infinity'. To understand this properly we have to look at the action of G_c by Möbius maps on $\hat{\mathbb{C}}$.

Recall that the unit sphere boundary of the ball model of $\mathbb{H}^3$ is infinitely far in terms of hyperbolic distance from the centre, and can be identified with $\hat{\mathbb{C}}$.

Suppose G is a discrete group of Möbius maps. Then its action on $\mathbb{H}^3$ extends to an action on $\hat{\mathbb{C}}$, which as we have seen typically had a limit set Λ where the orbits accumulate. The complement of Λ -- the white part in this figure -- is called the *regular set*, denoted $\Omega = \Omega(G)$.

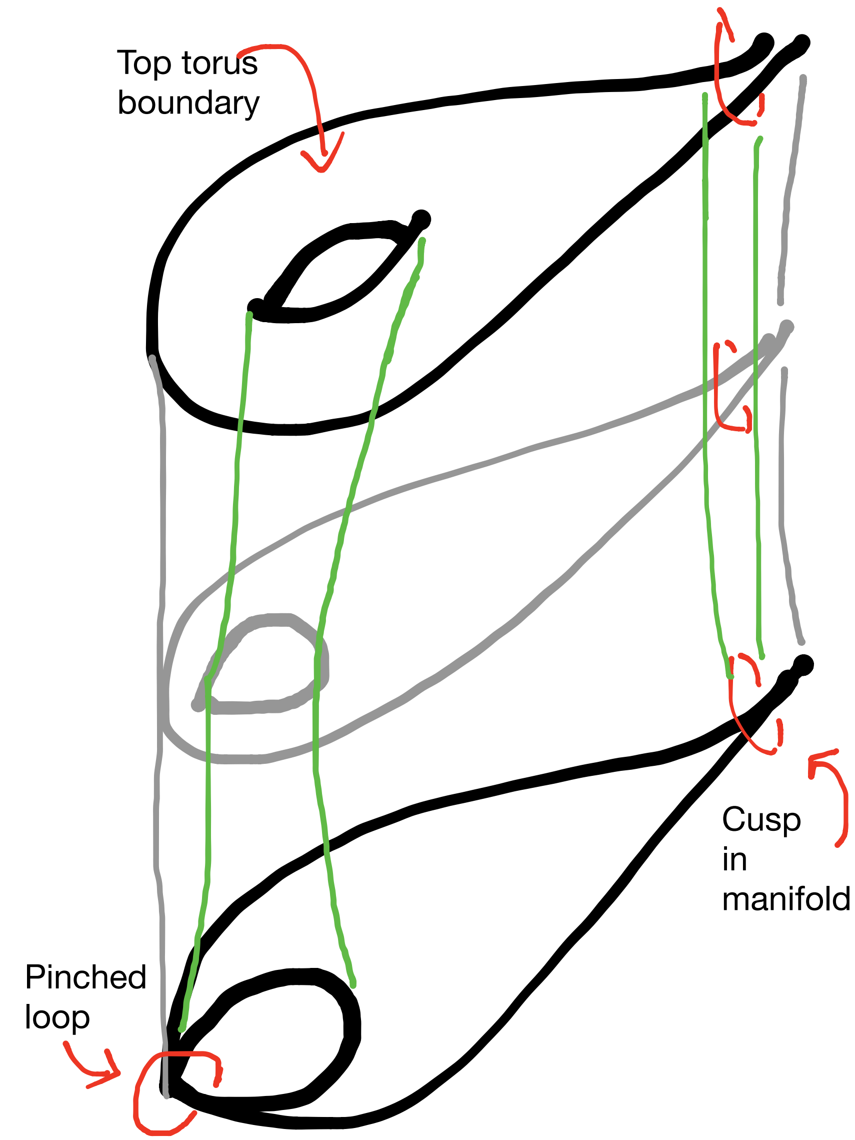
Limit set and regular set shown on the Riemann sphere. By David Wright.

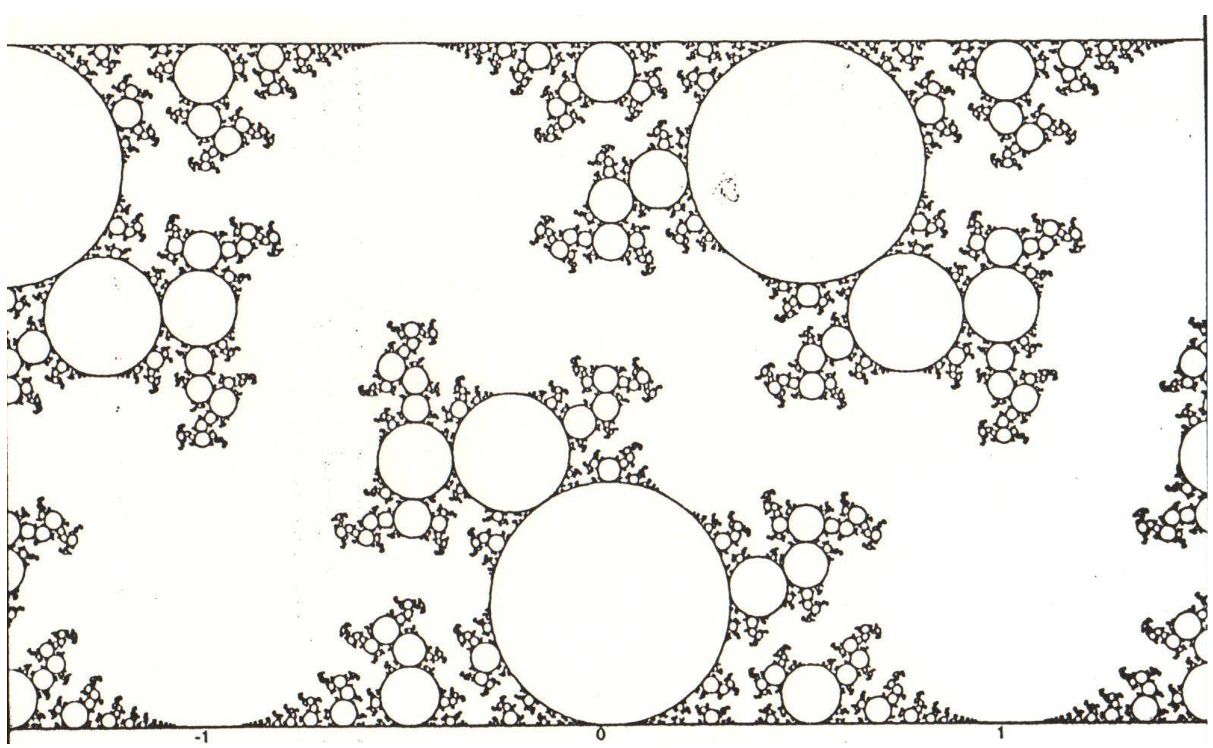
Since G acts properly discontinuously on Ω, it makes sense to identify all points in each G orbit to one point and form the quotient Ω/G. Since $\Omega \subset \mathbb{C}$, the space Ω/G inherits a complex structure from $\mathbb{C}$, equivalently the structure of a Riemann surface.

In general Ω may have many connected components. *Ahlfors finiteness theorem (1964)* states that Ω/G is a finite union of Riemann surfaces of finite type.

Mumford's problem: The Riemann surfaces

What are these Riemann surfaces in our case? The 'top' boundary of the hyperbolic manifold M is the punctured torus $T^* = S_{1,1}$ and the 'bottom' boundary is a sphere with 3 punctures $S_{0,3}$.





In this picture of a limit set for a free discrete group G_c, the regular set Ω consists of one simply connected component Ω_0 and lots of round disks D_j. Ω_0 is G invariant and $\Omega_0/G = T^*$.The disks D_j are all mapped to one another G and $D/G = S_{0,3}$.

Teichmüller theory and the set of free discrete groups $\mathscr{D}$

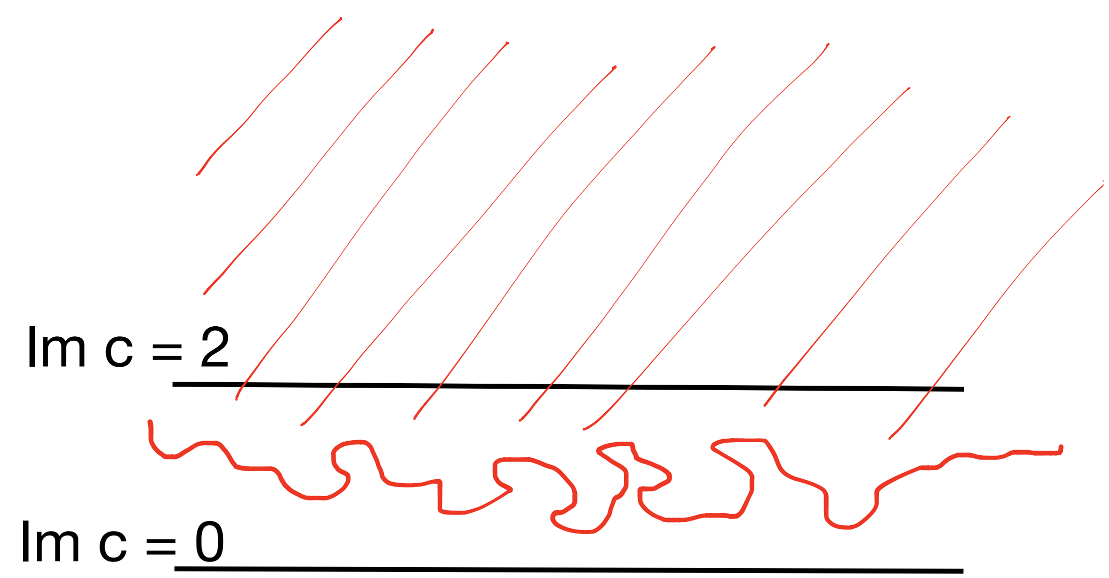
Teichmüller theory is the study of deformations of Riemann surfaces. The space of all complex structures on a given topological surface S is called its *Teichmüller space* $\mathscr{T}(S)$.

Let $\mathscr{D}$ denote the set of all free discrete groups G_c, so $\mathscr{D} \subset \mathbb{C}$. Mumford's problem is to find $\mathscr{D}$.

We can determine the rough shape of $\mathscr{D}$ from the Teichmüller theory of the two ends of $M = \mathbb{H}/G_c$, that is, of the Riemann surfaces $S_{0,3}$ and $S_{1,1}$.

It is well known that $\mathscr{T}(S_{0,3})$ is a point, that is, there is a unique complex structure on $S_{0,3}$.

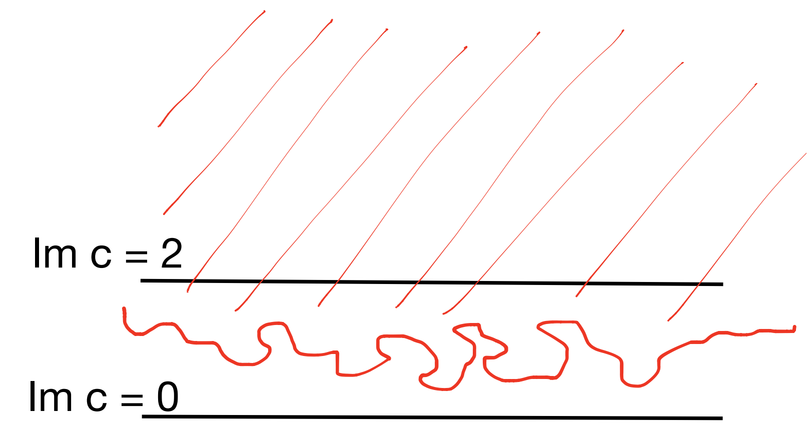
By theory due to Bers, $\mathscr{T}(S_{1,1})$ is conformally the upper half plane $\mathbb{H}$ — the same as the space of all conformal structures on a flat torus.



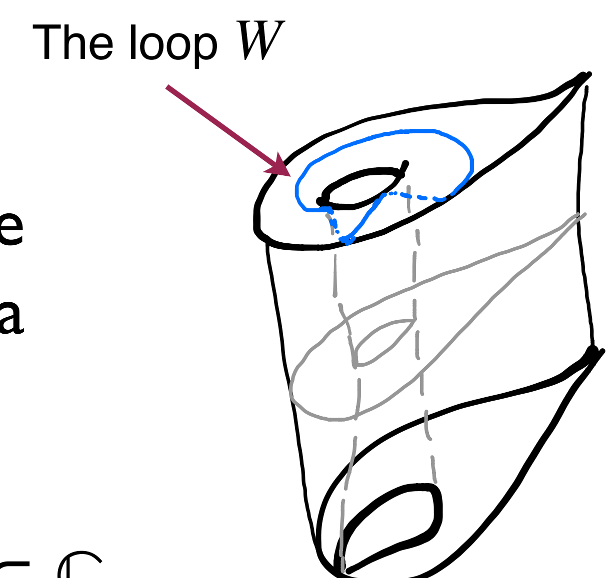
Mumford's problem is set up in such a way that the entire region above $\Im c = 2$ is in $\mathscr{D}$, and so that $\partial\mathscr{D}$ is between the horizontal lines $\Im c = 0$ and $\Im c = 2$.

Locating $\partial\mathcal{D}$: Cusp groups

Bers also described a special type of group, called *cusp groups*, on $\partial\mathcal{D}$. These are groups where an element in $G = \pi_1(T^*)$ which represents a simple curve on T^* is parabolic. Suppose $W \in G$ corresponds to a simple closed loop on the torus T^*. *Bers showed that there are points on $\partial\mathcal{D}$ for which $W = W_c$ is parabolic.*



Mumford realised cusp groups might be found computationally.



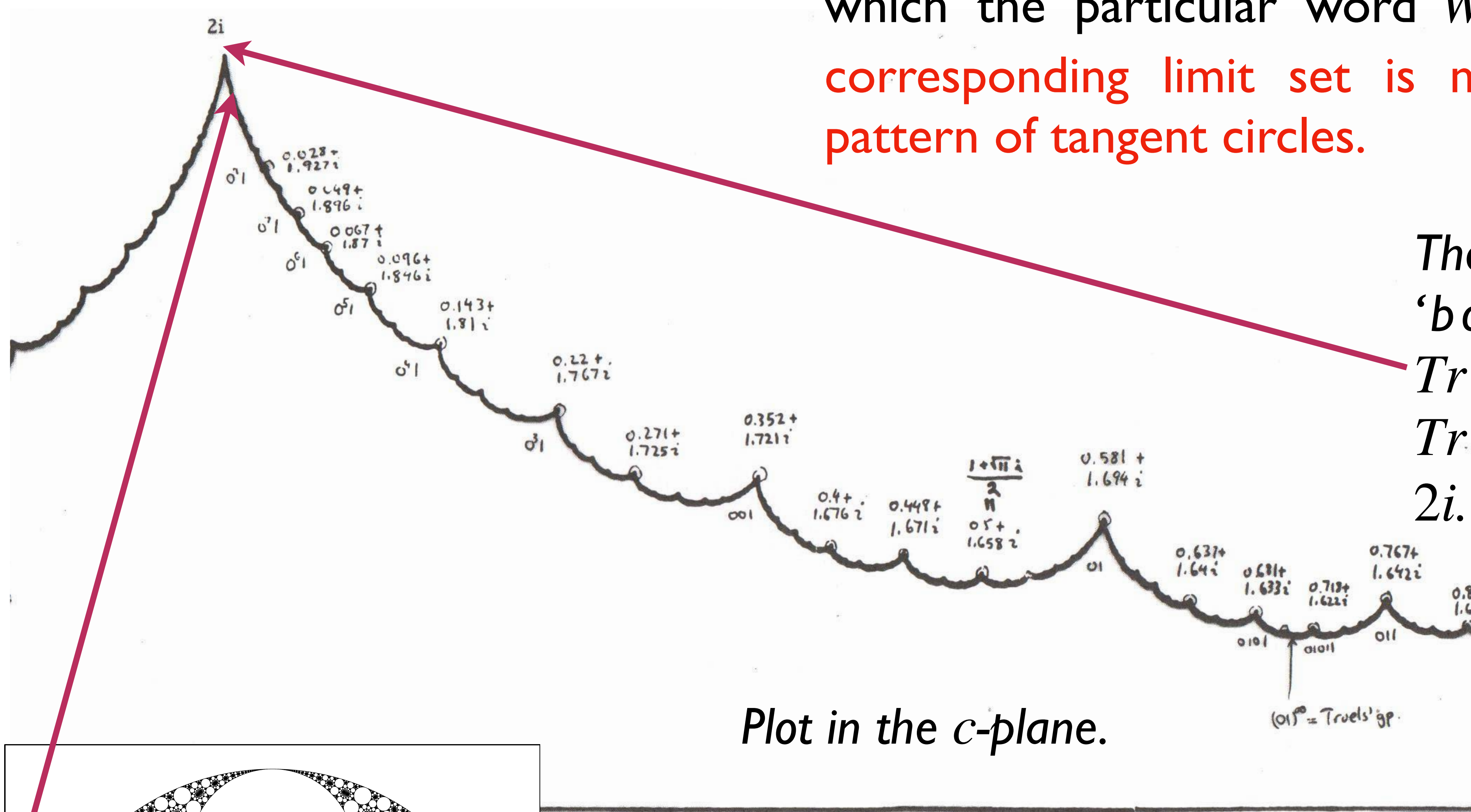
The possible free homotopy classes of loops on the torus, are enumerated by the rational numbers p/q. The word $W_{p/q}$ is a product of the two maps A, B: $W_{1/15} = A^{15}B, W_{2/5} = A^2BA^3B$.

The matrix coefficients of A and B depend on the parameter $c \in \mathbb{C}$, hence $Tr\,W$ is a polynomial in c. This gives a family of polynomial equations in c which can be solved to search for boundary points, for example at the $2/5$ boundary point $Tr\,W_{2/5} = Tr\,A^2BA^3B = \pm 2$.

Mumford and David Wright did a systematic search. Warning: There are lots of solutions for which W is parabolic but G_c is not discrete. These are no good!

First results

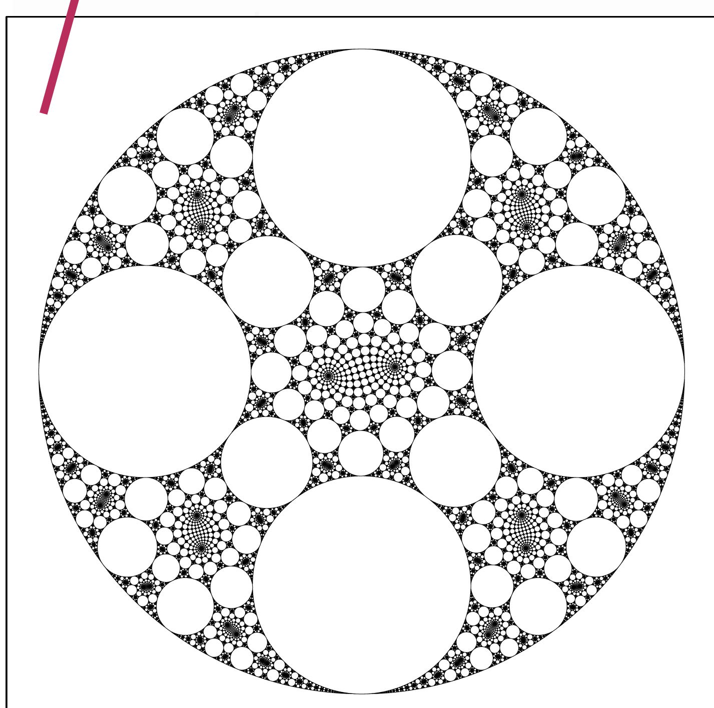
This is what they found.



Plot in the c-plane.

Each plotted point corresponds to a cusp group at which the particular word $W_{p/q}$ is parabolic. The corresponding limit set is made of a beautiful pattern of tangent circles.

The highest point on the 'boundary' is where $TrA = -2$. Since $TrA = ic$ this is the point $2i$.



The limit set corresponding to $1/15$ at which $A^{15}B$ is parabolic.

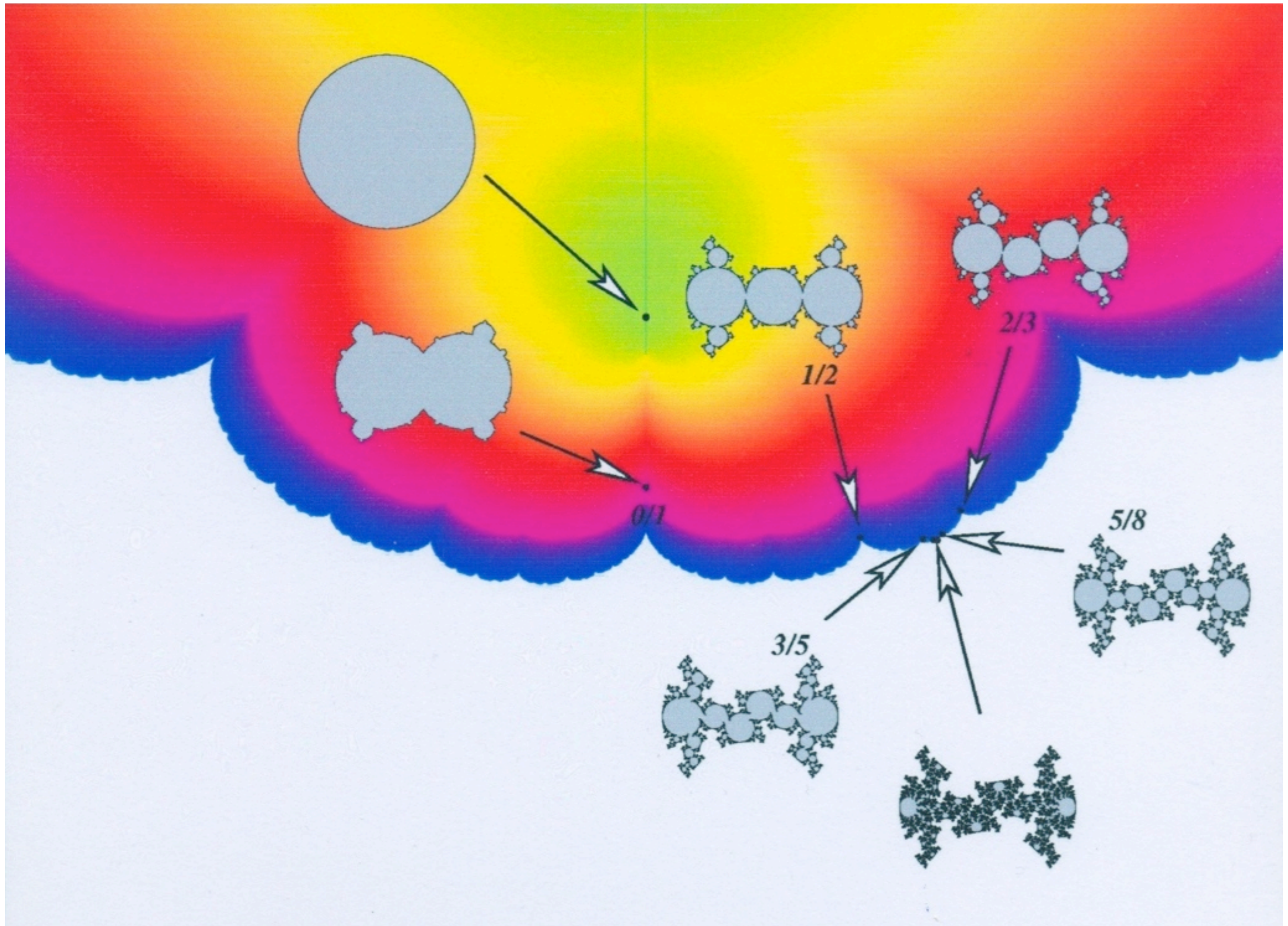
The collection of plotted points appear to form a curve which one could well believe is a dense subset of $\partial\mathscr{D}$. But is it?

Some theorems

Such pictures inspired some nice results, in particular a famous paper by Curt McMullen.

Theorem (McMullen 1991)
Cusp groups are dense on the boundary.

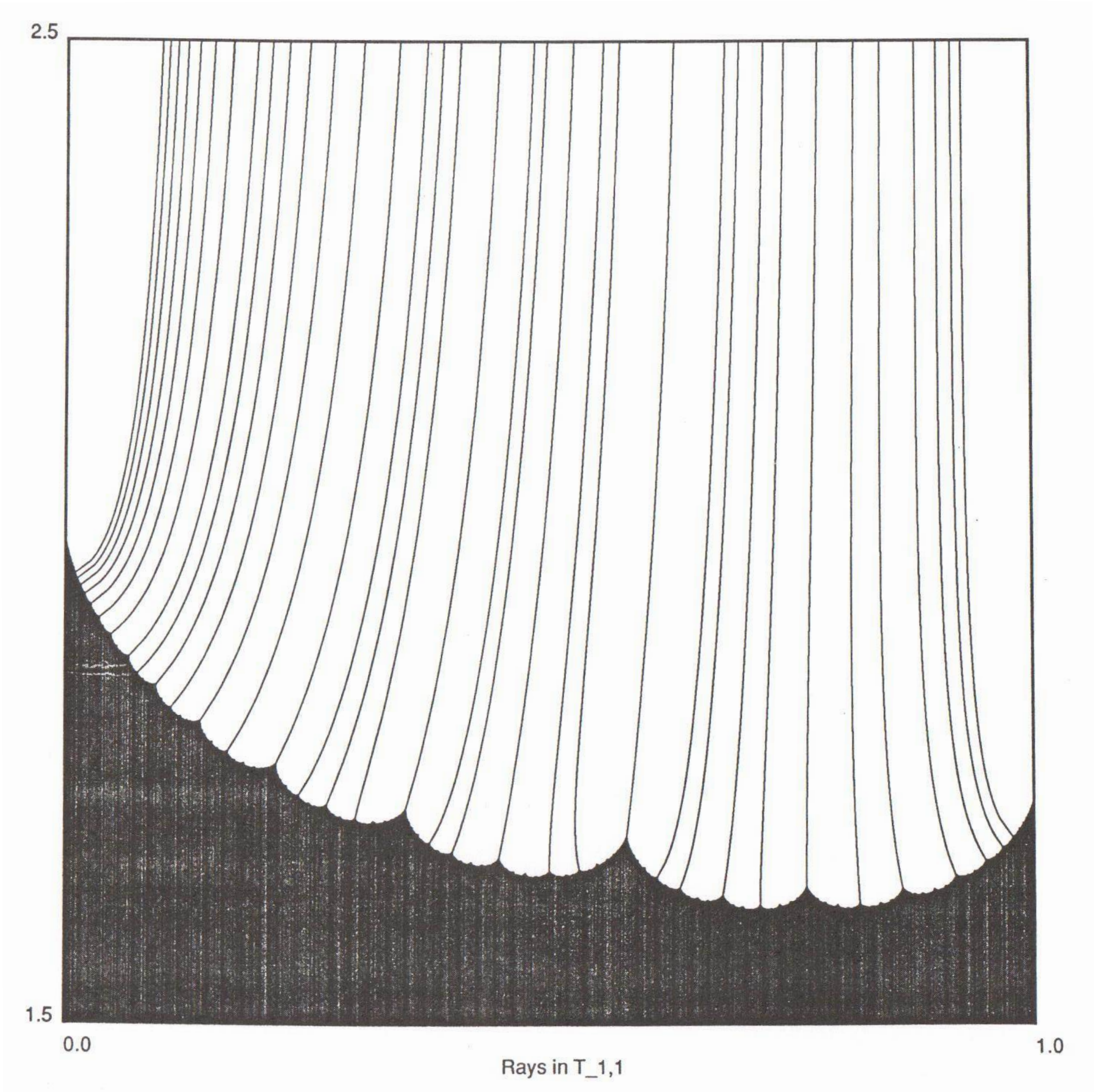
Theorem (Keen, Maskit, S. 1993)
The limit set of each cusp group is indeed formed by a pattern of tangent circles. There is a unique pattern and a unique cusp group for each rational $p/q \in \mathbb{Q}$.



This picture made later by David Wright shows a slightly different family in which $Tr\,A = c, Tr\,B = 3$. The picture is in the plane $Tr\,B = 3$. Limit sets for some special values of $Tr\,A$ are inserted. The coloured region indicates the free discrete groups. Notice how the limit sets of groups on or near the boundary contain lots of circles.

The rays

Going back to Mumford's problem, in 1990, Linda Keen and I asked David Wright to plot the curve starting at the point $Tr\,W_{p/q} = 2$ and *continuing along the locus where* $Tr\,W_{p/q} > 2$. The results were astonishing.



By David Wright, 1990

Linda Keen, CUNY

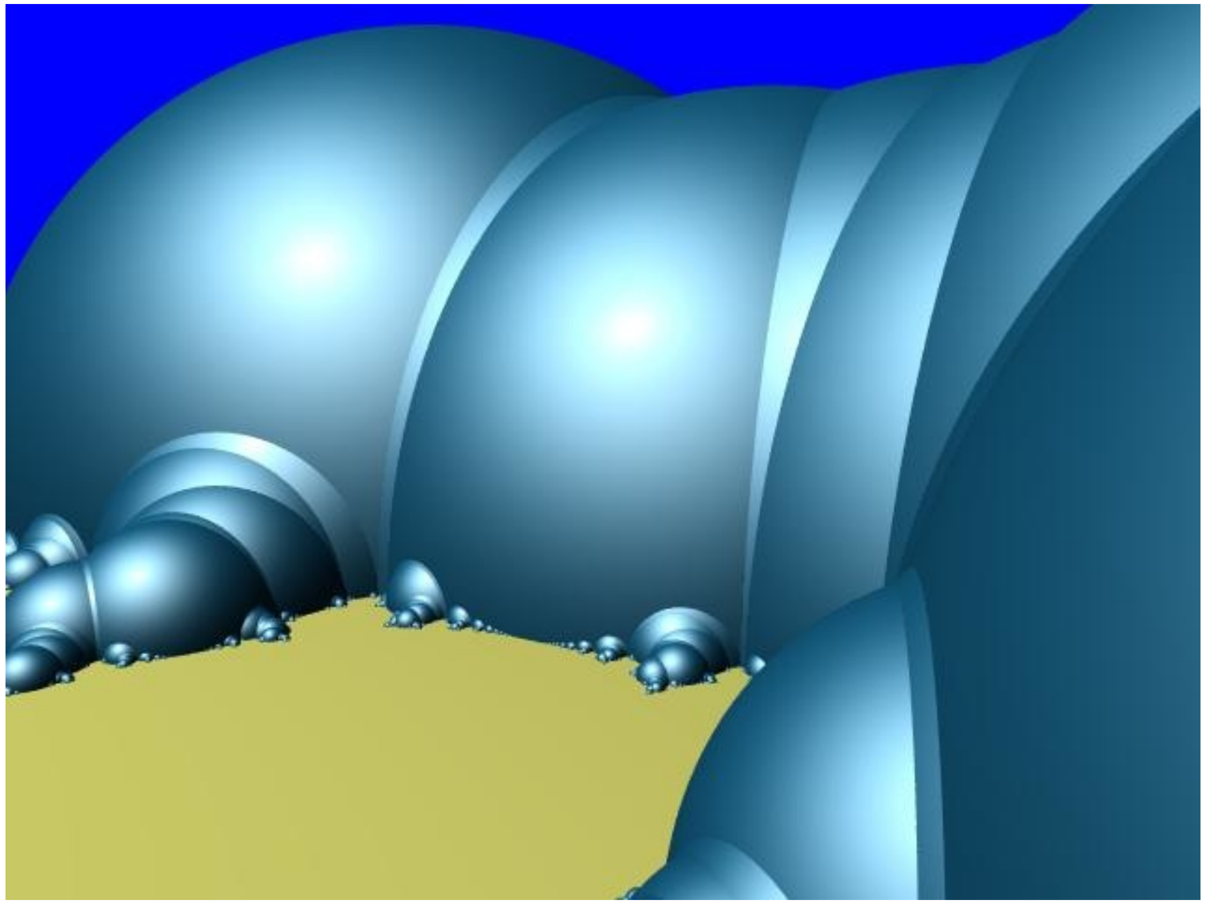
There is one ray for each rational p/q. The ray starts at the boundary point where $Tr\,W_{p/q} = 2$ and follows the path along which $Tr\,W_{p/q}$ increases through real values to ∞.

The p/q- ray is asymptotic to the line $\Re c = 2p/q$.

What are these rays and do they really behave as they seem to?

The meaning of the rays: moving into 3D

To understand the rays, it turned out to be necessary to move into $3D$ and use some of Thurston's ideas. The key is to look at the *hyperbolic convex hull* $\mathscr{C}$ of the limit set.



Picture by Yair Minsky

Join any two points in the limit set by a hyperbolic geodesic. The resulting $3D$-object is invariant under the group.

The boundary $\partial\mathscr{C}$ of $\mathscr{C}$ is made up of pieces of hyperbolic planes, glued together along bending lines which are themselves hyperbolic lines.

Thurston introduced the study of 'surfaces' like this, which he called *pleated surfaces*. They play a crucial role in many subsequent developments.

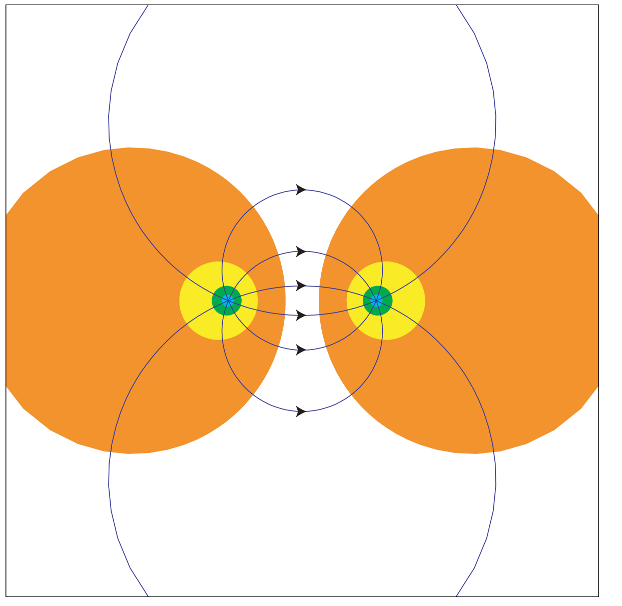
Interpretation in the limit set

Considered as a $3D$ isometry, the *axis* of the Möbius map $z \mapsto (az+b)/(cz+d)$ is the hyperbolic line joining its two fixed points.

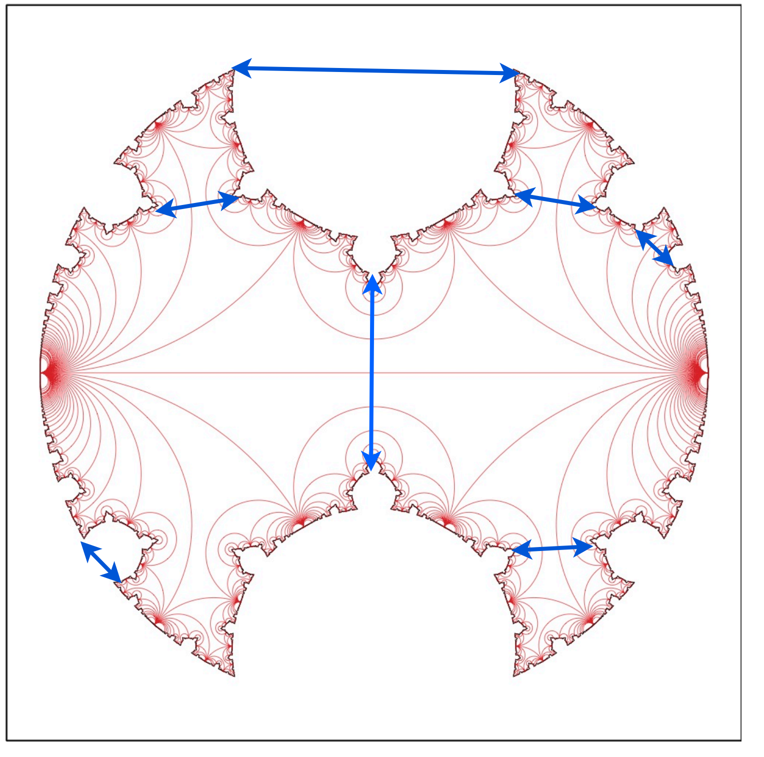
If the trace $a+d$ is real valued there is no spiralling, so no twisting round the axis in 3D-hyperbolic space.

Suppose $\partial\mathscr{C}$ has a bending line which is the axis of a $3D$ isometry. Then there is no twisting about its axis. *This implies that the trace of the corresponding matrix is real valued.* This was the key to the problem.

Along a ray, a particular element $W \in G_c$ has real trace and the axis of W and its conjugates form the bending lines.



A Möbius map with real trace.



In this limit set there are lots of overlapping circles which make planes in $\partial\mathscr{C}$. The hyperbolic lines joining the points where the circles overlap form the bending lines.

Interpretation of the rays

Along a ray, the axis of W_c and its conjugates form the bending lines of $\partial\mathscr{C}$. These bending lines project to a collection of disjoint geodesics (closed or open) on T_c^*. (Note that $T_c^* \subset \partial\mathscr{C}$ can be identified as a pleated surface with its own hyperbolic structure.)

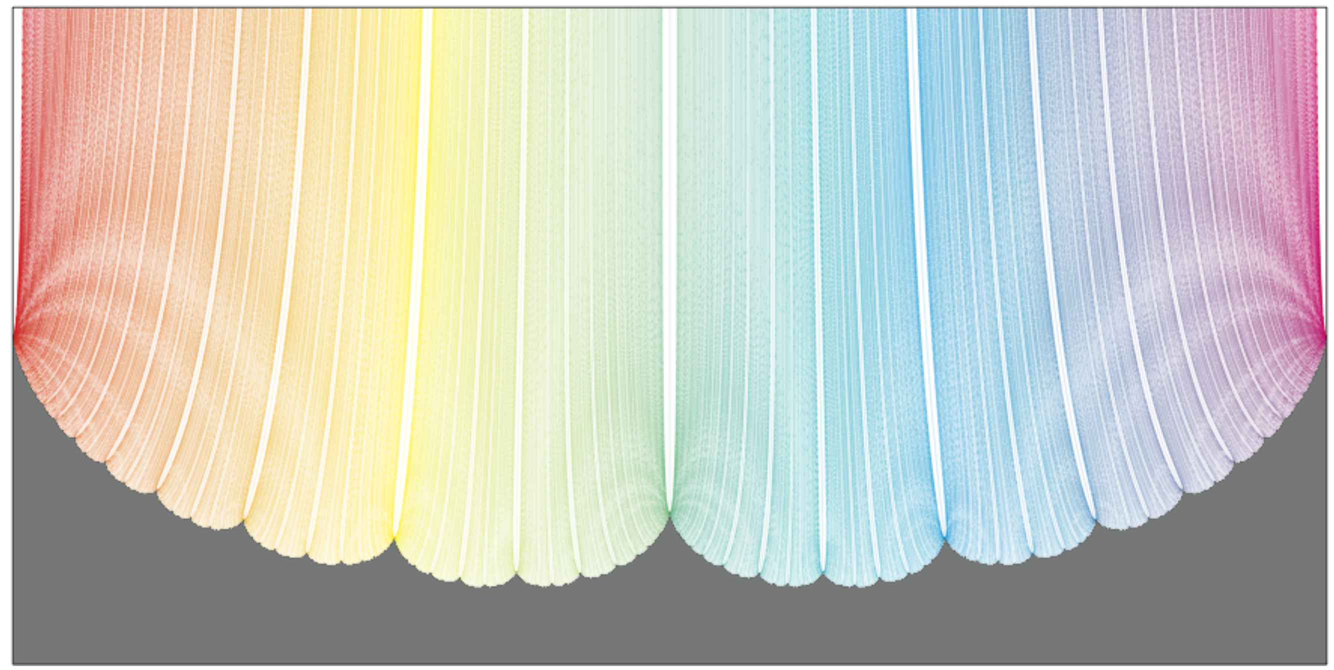
This key observation allowed us to prove all the properties of the rays seen in the pictures.

- There is one ray for each possible closed geodesic on T^*. No two rays can intersect because on T^* any two closed geodesics intersect, and this would mean two intersecting bending lines.
- There is one cusp group, and hence one ray, for each p/q, corresponding to the possible simple closed curves on T^*. At any point on the ray, there is no spiralling around the axis of $W_{p/q}$ so $Tr\, W_{p/q} \in \mathbb{R}$.
- Each ray starts at the cusp group on $\partial\mathscr{D}$ where the curve $\gamma_{p/q}$corresponding to $W_{p/q}$ has zero length. The length of $\gamma_{p/q}$, and hence $Tr\, W_{p/q}$, increases monotonically from zero to ∞.
- If the bending lines do not project to a simple closed curve, then they project to what is called a *geodesic lamination* on T^*. These interpolate the rationals in the obvious way.

Some theorems

The Mumford-Wright picture of the boundary has led to many nice theorems. Our ray picture proves that the boundary is indeed located as claimed.

Theorem (Keen-S.1994) The function $Tr\,W_{p/q} \in \mathbb{R}$ has no singularities along its ray. It is asymptotic to the vertical line $\Re c = 2p/q$. The rays fill out $\mathscr{D}$ densely and are interpolated by irrational rays which also have a geometric meaning.



Picture by David Wright

Theorem (Minsky, 2001)
Ending lamination theorem for T^*.
Each irrational ray ends in a unique point. In consequence the boundary $\partial\mathscr{D}$ is a Jordan curve.

Theorem (Many authors c.2000-05)
Bers density theorem for T^*.
There are no discrete free groups G_c outside the region covered by the rays.

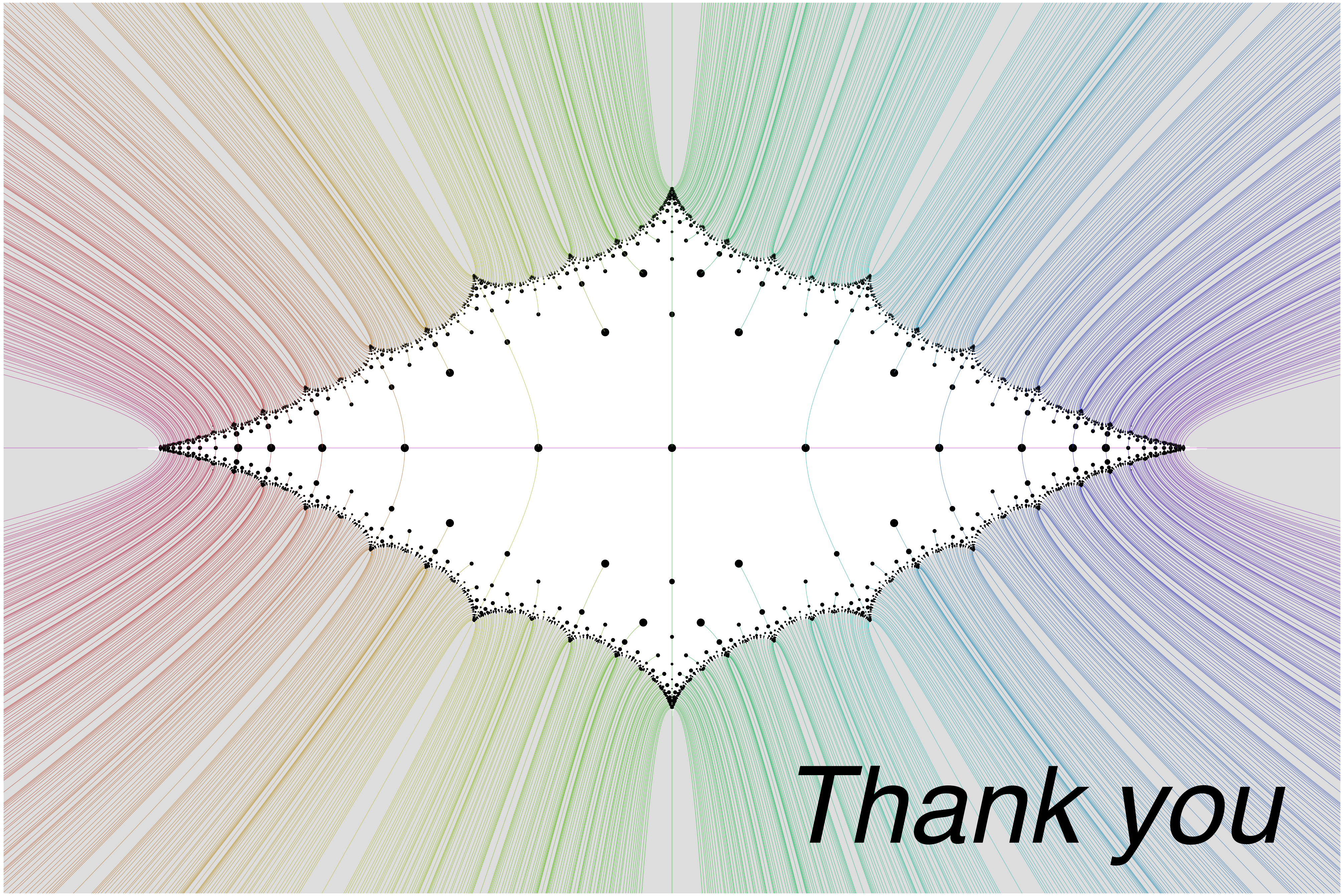
Other families, more parameters

In subsequent work over many years I have refined and extended these results to other one dimensional situations and also to surfaces of higher genus which involves parameter spaces of more complex dimensions and which are much more complicated.

I have worked on this topic with a number of other people including (in roughly chronological order)
Yohei Komori
Raquel Díaz
John Parker
Young Eun Choi
Sara Maloni
Ser Peow Tan
Yasushi Yamashita

Hideki Miyachi also contributed to the study of the boundary. Masaaki Wada has made beautiful pictures, as has Kentaro Ito. Ken'ichi Oshika has made many important contributions to the general theory of discrete groups.

Currently, inspired by Makoto Sakuma, I am looking at what happens along the extensions of the rays out side the parameter space — it seems probable that all discrete groups including knot groups are located along these rays.



Picture showing rays and extended rays in another one-parameter family called the Riley slice. The black dots represent discrete but non-free groups. Picture by Yasushi Yamashita.

Credits: Unless otherwise noted, the pictures are made by Mumford and Wright and most are in Indra's Pearls.