An introduction to embedding problems in symplectic geometry

Dusa McDuff

Department of Mathematics, Barnard College, Columbia University

dusa@math.columbia.edu

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Plan of lecture:

A symplectic structure is a rather elusive geometric structure that can be put on an even dimensional space. After a brief introduction I will explain some recent developments in the symplectic embedding problem.

- (I): Very brief introduction to symplectic geometry
- (II) The symplectic embedding problem
- (III) The Fibonacci staircase

What is symplectic geometry?

Euclidean (or Riemannian) geometry makes measurements using a symmetric bilinear form, such as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots x_n y_n, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Symplectic geometry makes measurements using a skew symmetric bilinear form, such as

$$\omega_0 = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n} \text{ on } \mathbb{R}^{2n}.$$

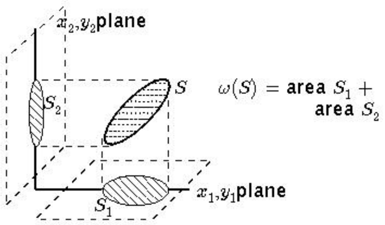
More generally, a symplectic form ω on a (necessarily even dimensional) manifold is a differential 2-form ω that locally looks like ω_0 with respect to suitable local coordinates.

Thus in 2-dimensions, ω is an area form, and symplectic geometry is the study of area preserving diffeomorphisms. These have many interesting properties (e.g. more fixed points than an arbitrary diffeomorphism). Arnol'd realised that in higher dimensions these properties do *not* hold for volume preserving diffeomorphisms, but he conjectured that they do hold for symplectic diffeomorphisms. His conjectures have inspired much work.

The standard symplectic form ω_0 in $\mathbb{R}^4$

$\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2$,
a sum of area forms

S is a piece of surface that you project in two different ways and then add the areas.



Since ω_0 is closed, i.e. $d\omega_0 = 0$, we get flabby measurements: by Stokes' theorem, the area of a surface S can be written as an integral $\int_S \omega_0$; that does not change as S moves as long as the boundary remains fixed. In physics, the pair x_i, y_i represents the position and velocity of a particle in one direction – so a particle moving in 3-space gives 6 coordinates. The symplectic form in $\mathbb{R}^6$ gives an important (but geometrically somewhat obscure) measurement of the mutual entanglement of position and velocity.

Many symplectomorphisms:

In general, a symplectic structure on a $2n$-dimensional manifold M is a closed, nondegenerate 2-form ω.

Every function $H : M \to \mathbb{R}$ generates a flow $\phi_t^H, t \in \mathbb{R}$ (1-parameter group of motions of space) that preserves ω: $(\phi_t^H)^*(\omega) = \omega$. Such transformations are called symplectomorphisms.

The flow is a solution of Hamilton's differential equations — generated by the vector field X_H given by $\omega(X_H, \cdot) = dH(\cdot)$ – so that in $\mathbb{R}^{2n}$ we have $\frac{\partial x}{\partial t} = \frac{\partial H}{\partial y}, \frac{\partial y}{\partial t} = -\frac{\partial H}{\partial x}$.

Example: if $H = \frac{1}{2}(x^2 + y^2)$ on $(\mathbb{R}^2, dx \wedge dy)$, we find

$$dH = xdx + ydy \implies X_H = y\partial_x - x\partial_y$$

giving a clockwise rotation, preserving the circles $H = const$.

But there are many, much more twisty symplectomorphisms Because there are so many, symplectic geometry is very flexible.
But it also displays interesting rigidity

Symplectic embeddings

In 1985 Gromov asked:
what can be said about the image $\phi(B^{2n})$ of a ball under a symplectomorphism ϕ of $\mathbb{R}^{2n}$?

In 2-dimensions nothing interesting happens:

(Moser - 1965) *If a closed disc $D \subset \mathbb{R}^2$ is diffeomorphic to a closed region U of the same total area, there is an area preserving diffeomorphism $\phi : D \overset{\cong}{\to} U$.*

But higher dimensions are very interesting. Consider the ball

$$B^{2n}(a) = \{(z_1, \ldots, z_n) : \pi(|z_1|^2 + \cdots + |z_n|^2) \le a\} \subset \mathbb{C}^n = \mathbb{R}^{2n}$$

and the cylinder $Z(A) = \{(z_1, \ldots, z_n) : \pi|z_1|^2 \le A\} \subset \mathbb{R}^{2n}$.

Nonsqueezing Theorem (Gromov: 1985) *There is a symplectic embedding $B^{2n}(a) \overset{s}{\hookrightarrow} Z(A)$ if and only if $a \le A$.*

The volume preserving map $(z_1, \ldots, z_n) \mapsto (\lambda z_1, \frac{1}{\lambda} z_2, z_3, \ldots, z_n)$,
$\lambda := \sqrt{\frac{A}{a}}$ squeezes the ball into the cylinder.

Symplectic capacities:

Although the Nonsqueezing Theorem may seem just like a curiosity, it turns out to be a cornerstone of the modern theory.

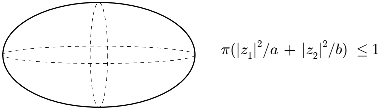
Gromov, Ekeland–Hofer (1980s): Given an open $U \subset \mathbb{R}^{2n}$ define the symplectic capacity by

$$c(U) = \sup\{a : B^{2n}(a) \text{ embeds symplectically in } U\}.$$

- $c(U)$ is a symplectic invariant;
- it is essentially 2-dimensional, eg the cylinder $Z(A)$ is a set of infinite volume with finite capacity
- any orientation preserving diffeomorphism ϕ that preserves c (i.e. $c(\phi(U)) = c(U)$ for all U) is an (anti)-symplectomorphism, i.e. $\phi^*(\omega) = \pm\omega$. So: it characterizes symplectomorphisms
- What other obstructions are there to symplectic embeddings?

Embedding 4-dimensional Ellipsoids

Let $E(a,b)$ be the ellipsoid $\{(z_1,z_2) : \pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) \le 1\}$.



the ellipsoid $E_{a,b}$

Hofer conjectured around 2010 that $\mathrm{int}E(a,b)$ *embeds symplectically in* $\mathrm{int}E(c,d)$ *if and only if* $\mathcal{N}(a,b) \le \mathcal{N}(c,d)$. Here $\mathcal{N}(a,b)$ is the set of all numbers $ka + \ell b,\ k,\ell \ge 0,$ arranged with multiplicities in increasing order.So,

$\mathcal{N}(2,2) = (0, \underbrace{2,2}, \underbrace{4,4,4}, \underbrace{6,6,6,6}, \underbrace{8,8,8,8,8}, \dots),$ and

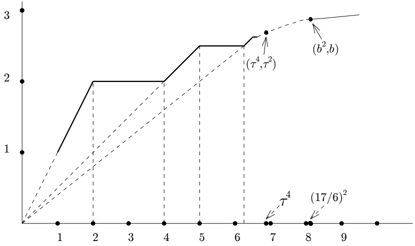
$\mathcal{N}(1,4) = (0, \underbrace{1,2}, \underbrace{3,4,4}, \underbrace{5,5,6,6}, \underbrace{7,7,8,8,8}, \dots$ Thus $\mathcal{N}(1,4) \le \mathcal{N}(2,2)$ because the first sequence is termwise no larger than the second.

These numbers are the actions of the ECH generators: ECH = embedded contact homology — a 4-dimensional Floer-type homology theory related to gauge theory; whose generators are unions of closed orbits of the boundary Hamiltonian flow.

This conjecture now proved by McDuff (2012). An illustration of what it means:

The "ellipsoid into ball" embedding capacity

For $x \geq 1$ define $c(x) := \inf\{\mu : E(1,x) \text{ embeds sympl. in } B^4(\mu)\}$.
This function was calculated by McDuff–Schlenk (2012).



- For $x < \tau^4 \approx 6.7$ (where $\tau = \frac{1+\sqrt{5}}{2}$) there is an infinite staircase (with numerics based on the Fibonacci numbers),
- for $x \geq 8\frac{1}{36} = \left(\frac{17}{6}\right)^2$, $c(x) = \sqrt{x}$ – no obstruction except for volume
- $\tau^4 < x < 8\frac{1}{36}$ is a transitional region;
- there are rather few results or plausible guesses as to behavior in dim > 4.
  The obvious analog of Hofer's conjecture is false (Guth) since
  $E(1,S,S) \overset{s}{\hookrightarrow} E(3+\varepsilon, 3+\varepsilon, S^3)$ for all S

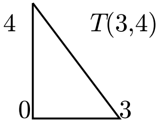
Connection with number theory via Toric models 13.

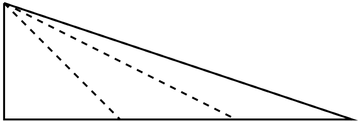
Using coordinates ($t = \pi|z|^2, \theta = \arg z$) and forgetting θ gives a map

$$\mathbb{C}^2 \to \mathbb{R}^2, \quad (z_1, z_2) \mapsto (t_1, t_2) := (\pi|z_1|^2,\ \pi|z_2|^2).$$

It takes the ellipsoid $E(a, b) = \left\{\pi\frac{|z_1|^2}{a} + \pi\frac{|z_2|^2}{b} \le 1\right\}$ to the triangle

$$T(a, b) := \ \left\{(t_1, t_2) \in \mathbb{R}^2_+ :\ 0 \le t_1, t_2, \frac{t_1}{a} + \frac{t_2}{b} \le 1\right\}.$$





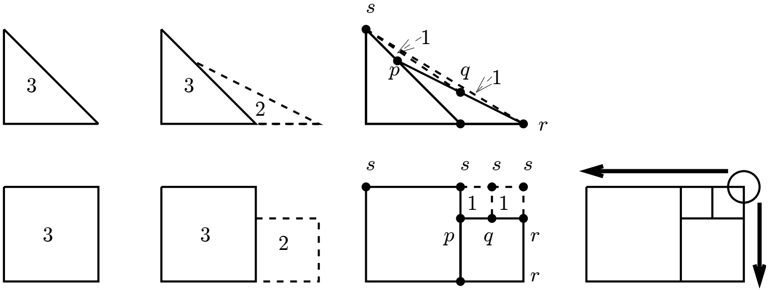
The ball $B(1)$ maps to the standard triangle $T(1, 1)$, with various affine equivalent images given by integral changes of basis of $\mathbb{C}^2$ (eg use $(z_1, z_1 + z_2)$). Thus $T(1, 3)$ can be cut into three standard triangles; cf. diagram on right.

Hence $E(1, 3)$ contains three disjoint balls $B(1) \sqcup B(1) \sqcup B(1)$.

General Triangle decompositions

In fact any triangle $T(a, b)$ with $a, b \in \mathbb{Z}$ can be decomposed into standard $\triangle$s of different sizes, (a standard triangle is linearly equiv to some $T(c, c)$)

On the top: we build $T(5, 3)$ from $T(3, 3) \sqcup T(2, 2) \sqcup T(1, 1) \sqcup T(1, 1)$.

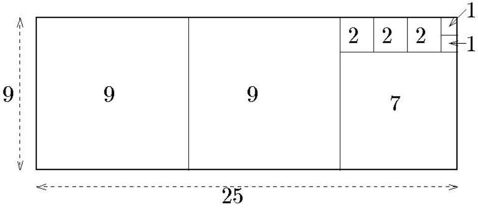


On bottom: we build a rectangle from squares. Combinatorially these decomp are same, but the second is much easier to understand. To get the triangles from the rectangles, remove the top right point and collapse top and right sides to points in affine way.

Continued fractions and decompositions

The combinatorics of this decomposition of a triangle into standard triangles (or of a rectangle into squares), is the same as that governing the weight expansion $W(\frac{p}{q})$ of a rational fraction $\frac{p}{q}$. e.g.

$W(\frac{25}{9}) = (\underbrace{9,9},7,\underbrace{2,2,2},\underbrace{1,1})$



corresponds to

$$\frac{25}{9} = [2;1,3,2] = 2 + \frac{1}{1+\frac{1}{3+\frac{1}{2}}} = 2 + \frac{1}{1+\frac{2}{7}} = 2 + \frac{7}{9}.$$

The entries of the continued fraction are the multiplicities of the weights

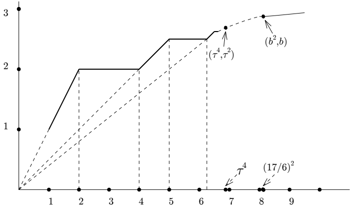
Embedding ellipsoids vs disjoint balls.

Key Points: Suppose that $\frac{p}{q}$ has normalized weight expansion $w(\frac{p}{q}) = (w_1, w_2, \ldots, w_n)$ (where $w_1 = 1$). Then

- $E(1, \frac{p}{q})$ embeds into $B^4(\mu)$ if and only if one can embed the disjoint union of balls $\sqcup_{i=1}^n B(w_i)$ into $B^4(\mu)$.
- $\sqcup_{i=1}^n B(w_i)$ embeds into the ball $B^4(\mu)$ if and only if there is a symplectic form on the n-fold blow up of $\mathbb{C}P^2(\mu)$ in the cohomology class $\alpha := \mu\, PD(L) - w_1\, PD(E_1) - \cdots - w_n PD(E_n)$ ($PD(E_i)$ is the Poincaré dual to the exceptional divisor given by the ith blow up)
- (Li-Li, 1990s) this happens if and only if
  - $\alpha^2 > 0$ (volume constraint)
  - $\int_E \alpha > 0$ for all exceptional divisors in the blow up.

  So we now have a geometric/algebraic problem — what are the homology classes of the relevant exceptional divisors? In the case when we map into a ball, their coefficients come from Fibonacci numbers.

Fibonacci staircase revisited



- We found that the exceptional classes in blow ups of $\mathbb{C}P^2$ that gave the sharpest obstruction to embedding $E(p, q)$ had the form $dL - \sum m_i E_i$,where $(m_i) = W(p/q)$ and d, p, q are odd placed Fibonacci numbers. These are called perfect classes.
- Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- For example at $p/q = 2/1$ with $W(2) = (1^{\times 2})$ the obstruction is $L - E_{12} = L - E_1 - E_2$ (blow up of a line through 2 generic points.
- at $p/q = 5/1$ with $W(5) = (1^{\times 5})$ the obstruction is $2L - E_{1\dots5}$ (blow up of a conic through 5 generic points.
- at $p/q = 13/2$, we have $W(13/2) = (2^{\times 6}, 1^{\times 6})$ and the obstructive class is $5L - 2E_{1\cdots6} - E_{78}$.
- Recently I (and various collaborators) have been studying the staircases that exist for embeddings ellipsoids into Hirzebruch surfaces (one point blow up of $\mathbb{C}P^2$). We found a very interesting fractal pattern.

A very few references:

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