Mathematical reflections on locality

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I. The principle of locality revisited

The principle of locality states that an object is influenced directly only by its immediate surroundings.

Thus, one can separate events located in different regions of space-time and should be able to measure them independently.

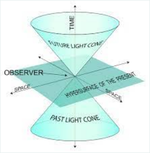
Our aim

- Propose a mathematical framework which encompasses the main features of the locality principle in QFT;
- use this framework to carry out renormalisation (evaluate meromorphic germs at their poles) in accordance with the locality principle.

Causal separation

Light cone, past and future

In the Minkowski space $(\mathbb{R}^d, g)$, where $g(x,y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ is the Lorentzian scalar product, there is a notion of "past" and "future":



(picture downloaded from Wikipedia)

Two sets S_1 and S_2 are causally separated $(S_1 || S_2)$ if and only if S_i does not lie in the future of S_j for $i \neq j$.

Locality in axiomatic QFT

The Wightman field $\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathcal{O}(H)$ obeys the locality axiom

$$\mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \Longrightarrow [\varphi(f_1), \varphi(f_2)] = 0. \tag{1}$$

The (relative) scattering matrix S_f satisfies the locality condition

$$\begin{aligned} \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2) \quad &\Longrightarrow \quad S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2) \\ &\Longrightarrow \quad [S_f(f_1), S_f(f_2)] = 0. \end{aligned} \tag{2}$$

Mathematical interpretation

We introduce two binary relations

- on sets:

$$O_1 \top' O_2 \Leftrightarrow [O_1, O_2] = 0, \tag{3}$$

- on test functions:

$$f_1 \top f_2 \Leftrightarrow \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2). \tag{4}$$

Interpretation of (1) as a locality map (see later)

$$f_1 \top f_2 \Longrightarrow \varphi(f_1) \top' \varphi(f_2). \tag{5}$$

Interpretation of (2) as a locality morphism (see later)

$$f_1 \top f_2 \Longrightarrow S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2). \tag{6}$$

II. Locality as a symmetric binary relation

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set.

$$x_1 \top x_2 \iff (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top Y \iff X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \iff X \perp Y$ on subsets X, Y of an euclidean vector space V.

(almost-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions $\phi, \psi \in \mathcal{D}(U)$ are independent i.e., $\phi \top \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \iff P(A \cap B) = P(A)\, P(B).$$

Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$$L_1 \top L_2 \iff L_1 \pitchfork L_2 \iff T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2.$$

Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \iff m \wedge n = 1.$$

Partial products

- Locality set: $(X, \top)$,
- Polar set: $U^\top := \{x \in X, x \top u \quad \forall u \in U\}$ for $U \subseteq X$;
- Graph of the locality relation: $\top = \{(x_1, x_2) \in X^2, \quad x_1 \top x_2\}$;
- Partial product: $m_X : X \times X \supset \top \longrightarrow X$ i.e. $m_X(\top) \subset X$.

$(X, m_X, \top)$ locality semi-group

semi-group condition: $\forall U \subseteq X, \quad m_X\left((U^\top \times U^\top) \cap \top\right) \subseteq U^\top$
or equivalently

$$(x_1 \top u_1 \text{ and } x_2 \top u_2 \quad \forall u_1, u_2 \in U) \Longrightarrow (m_X(x_1, x_2) \top w \quad \forall w \in U).$$

Counterexample

Equip $\mathbb{R}$ with the locality relation $x \top y \iff x + y \notin \mathbb{Z}$.

$(\mathbb{R}, \top, +)$ is NOT a locality semi-group: for $U = \{1/3\}$ we have $(1/3, 1/3) \in (U^\top \times U^\top) \cap \top$ but $1/3 + 1/3 = 2/3 \notin U^\top$

Locality category

Locality structures

- set X $\rightsquigarrow$ locality set $(X, \top)$;
- semi-group (X, m_X) $\rightsquigarrow$ locality semi-group $(X, m_X, \top,)$;
- vector space $(V, +, \cdot)$ $\rightsquigarrow$ locality vector space $(V, +, \cdot, \top)$
  $(U \subset V \Longrightarrow U^\top$ vector space);
- algebra $(A, +, \cdot, m_A)$ $\rightsquigarrow$ locality algebra $(A, +, \cdot, m_A, \top)$.

Locality morphisms: $f : (X, \top_X) \to (Y, \top_Y)$

- locality map:
  $(f \times f)(\top_X) \subset \top_Y$ or equivalently $x_1 \top_X x_2 \Longrightarrow f(x_1) \top_Y f(x_2)$;
- locality semi-group morphism $f : (X, m_X, \top_X) \to (Y, m_Y, \top_Y)$:

  f is a locality map such that
  $x_1 \top_X x_2 \Longrightarrow f(m_X(x_1, x_2)) = m_Y(f(x_1), f(x_2))$.

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III. Evaluating meromorphic germs at poles in QFT

Functions of several variables in QFT

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers Feynman amplitudes given by the coefficients of the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$*, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...*

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^\infty \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \quad h \text{ holom. at zero}$$

Questions:

1. How to evaluate f consistently at the poles $z_1 = \cdots = z_k = 0$?
2. What freedom of choice do we have for the evaluator?

Evaluating a fraction with a linear pole at zero

$$f(z_1, z_2) = \frac{z_1 - z_2}{z_1 + z_2}|_{z_1=0, z_2=0} = \begin{cases} 1\,? \\ 0\,? \\ 10000\,? \end{cases}$$

Speer's generalised evaluators

They consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (later called independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;
4. (continuity) If $f_n(\vec{z}_k) \cdot L_1^{s_1} \cdots L_m^{s_m} \xrightarrow[n\to\infty]{\text{uniformly}} g(\vec{z}_k)$ as holomorphic germs, then $\mathcal{E}_k(f_n) \underset{n\to\infty}{\longrightarrow} \mathcal{E}_k(\lim_{n\to\infty} f_n)$.

Drawback: Speer's approach depends on the choice of coordinates $z_1, \cdots, z_k, \cdots$.

IV. Locality on meromorphic germs comes to the rescue

Back to the locality principle in QFT

We consider $\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty}\mathcal{M}(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i \text{ linear in } z_1, \cdots, z_k, \quad h \text{ holom. at zero}$$

Aim: evaluate meromorphic germs at poles according to the principle of locality: "*two events separated in space can be measured independently*"

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in\mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \text{Meas}\ \underbrace{(a \vee b)}_{\text{concatenation}} = \text{Meas}(a) \cdot \text{Meas}(b).$$

- We shall later equip $\mathcal{M}$ with a **locality** relation $\top$;

Principle of locality revisited: locality evaluators

$f \top g \Longrightarrow \mathcal{E}(f \cdot g) = \mathcal{E}(f)\,\mathcal{E}(g)$ for two meromorphic germs f and g in an appropriate subalgebra $\mathcal{M}^\bullet$ of $\mathcal{M}$.

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).

Locality on meromorphic germs: orthogonality

- **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.
- Q inner product on $\mathbb{R}^k$ induces one on $\mathcal{L}(\mathbb{C}^k)$
- $f_1 \perp^Q f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.
- polar germs: $\mathcal{M}_-^{\bullet Q}(\mathbb{C}^k) \ni f \Longleftarrow h \perp^Q L_i$ for all $i = 1, \cdots, n$.
- Theorem: (L. Guo, S.-P., B. Zhang/ N. Berline, M. Vergne 2015) $\mathcal{M}^\bullet(\mathbb{C}^k) = \mathcal{M}_+(\mathbb{C}^k) \oplus^Q \mathcal{M}_-^{\bullet Q}(\mathbb{C}^k)$

Where we stand

Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- the Galois group $\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) isomorphisms of $(\mathcal{M}^\bullet, \perp^Q)$;
- $\mathcal{M}_-^{\bullet Q}$ is generated by polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Orthogonal projection

$\perp^Q$ induces a splitting

$$\mathcal{M}^\bullet = \mathcal{M}_+ \oplus^Q \mathcal{M}_-^{\bullet Q} \quad \text{and} \quad {\pi_+}^Q : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$$

Theorem [Guo, S.P., Zhang 2022]

Definition

A locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is a linear form which i) extends the ordinary evaluation* ev_0 *at zero and ii) factorises on independent germs (or is a locality character):*

$$f_1 \perp^Q f_2 \Longrightarrow \mathcal{E}(f_1) \perp^Q \mathcal{E}(f_2).$$

Example: Minimal subtraction scheme:

$\mathcal{E}^{\mathrm{MS}} : \quad \mathcal{M}^\bullet \xrightarrow{\pi_+^Q} \mathcal{M}_+ \xrightarrow{\mathrm{ev}_0} \mathbb{C}$ is a locality evaluator.

Theorem

Given an inner product Q*, a locality evaluator at zero* $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ *is of the form:* $\mathcal{E} = \underbrace{\mathrm{ev}_0 \circ \pi_+^Q}_{\mathcal{E}^{\mathrm{MS}}} \circ \underbrace{T_{\mathcal{E}}}_{\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)}$.

THANK YOU FOR YOUR ATTENTION!

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